Null structure groups in eleven dimensions

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Abstract

We classify all the structure groups which arise as subgroups of the isotropy group, $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$, of a single null Killing spinor in eleven dimensions. We construct the spaces of spinors fixed by these groups. We determine the conditions under which structure subgroups of the maximal null strucuture group $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ may also be embedded in $SU(5)$, and hence the conditions under which a supersymmetric spacetime admits only null, or both timelike and null, Killing spinors. We discuss how this purely algebraic material will facilitate the direct analysis of the Killing spinor equation of eleven dimensional supergravity, and the classification of supersymmetric spacetimes therein.

1 Introduction

Recent advances in the understanding of supergravity theories and their supersymmetric solutions have reached the stage where geometries solving the Killing spinor equation can be systematically classified and constructed. The key concept which renders this possible is that of a G-structure. In this work we undertake the task of finding all the possible structure groups defined by a set of Killing spinors, at least one of which is null, which can arise in eleven dimensional supergravity, and explicitly constructing the spaces

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of spinors that these groups preserve. Our motivation is ultimately to construct a road map to eleven dimensional null supersymmetry: we want a recipe to associate to any class of solution admitting a null Killing spinor, be it $M2$–branes with transverse $Spin(7)$ space, G_2 compactifications to four dimensions, more complex systems of intersecting membranes and so on, a well defined space of candidate Killing spinors and a set of rules to calculate all the restrictions given by supersymmetry on the geometry and the fluxes. The method that we will employ can be adapted without conceptual modifications to lower dimensional supergravities.

There are a number of reasons why we believe this undertaking to be of interest, of which we mention three in particular. First, a systematic understanding of the structure of supersymmetric solutions gives powerful tools for constructing new and interesting ones. An eloquent example is the discovery of black ring solutions in five dimensional supergravity [1], which has stimulated discussion on the non-uniqueness of supersymmetric asymptotically flat black-hole solutions and on M–Theory calculations of their entropy. Second, there is a natural desire to classify supersymmetric solutions, for example to understand which flux-compactifications to lower dimensions are admissible in string and M-theory [2]-[9], and more generally, to provide as detailed a map as possible of the entire landscape of supersymmetric spacetimes in M-theory. Third, there is the potential for advances in Riemannian geometry; for example, the construction of a new infinite family of Einstein-Sasaki manifolds has already emerged from the G-structure programme [10], [11].

Since we are going to employ a refinement of the standard G-structure approach, we will briefly review the development of the method itself. A direct precursor of the G–structure analysis is [12]. The utility of G-structures in the context of classifying supersymmetric spacetimes was first advocated and demonstrated in [13]. In [14] it was shown how powerful this technique can be when applied to simple lower dimensional supergravities. By now there exists an extensive literature on applications of G–structure techniques to supergravity [15]-[24]. The first papers to study eleven dimensional supergravity with G–structure methods are [25], [26]. There necessary and sufficient conditions are given on the metric and four–form for the existence of either a single timelike or a single null Killing spinor. The existence of such spinors is equivalent to the existence of an $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, respectively, and the constraints for supersymmetry are encoded in the intrinsic torsion. The drawback of the original approach was that it could be only used for the classification of spacetimes with minimal supersymmetry. However, following the suggestion of [25], a systematic and universally applicable refinement of the original method, capable of classifying all solutions admitting an arbitrary number of Killing spinors, was given in [27], and afterwards [28].

The contributions of [25, 26] represent a major theoretical advance because for the first time the study of supersymmetric solutions in eleven dimensions was based on a

systematic understanding of all the geometries that support a solution of the Killing equation. The only alternative known before (with the exception of spacetimes with maximal supersymmetry) was that of giving an ansatz for the solutions and then verifying supersymmetry a posteriori. Mathematically a G-structure on a spacetime M is a principal sub-bundle of the frame bundle of M. A spacetime has holonomy G if it admits a torsion-free G-structure. In more concrete terms, the deviation from holonomy of a supersymmetric spacetime induced by the presence of fluxes is encoded in the intrinsic torsion of the G-structure (a general discussion of G-structures is given, for example, in [29]). It is this that makes a G-structure such a useful concept in the classification of supersymmetric spacetimes.

However, the nature of eleven dimensional supergravity is such that the space of backgrounds on which the Killing equation can be solved is vast. Trying to start from any such general background and to impose the equations of motion of the theory on it seems at present a prohibitive task. A natural and physically interesting specialization to make is to the study of geometries with enhanced supersymmetry, where there are two or more linearly independent solutions of the Killing equation. Classifying all such geometries is what we mean by a refined G–structure classification. Attempting to do this in the original G-structure formalism, which worked at the level of the bilinears constructed from the Killing spinors, is technically very complicated. In [27] a new approach for solving this problem was proposed, and illustrated in seven dimensions, in which all calculations are performed on the spinors themselves rather than their bilinears (the formalism includes and generalises the algebraic Killing spinor techniques of [30]-[34]). Applying this method to any other supergravity is straightforward.

The essential point which makes calculations tractable is the simplified description of the spinors. It is shown in [27] that by acting on a fiducial spinor ϵ with matrices Q in an appropriate subset of the Clifford algebra, one may span the space of spinors, in any supergravity. One then decomposes the space of spinors into modules of the maximal compact factor of the isotropy group of the fiducial spinor ϵ, and likewise the spinorial basis into bases for the modules. In eleven dimensions, the appropriate groups are $SU(5)$ or Spin(7). The most efficient analysis of the Killing spinor equation then proceeds as follows, and may be broken into three phases.

First, one assumes that the fiducial spinor ϵ is Killing, and calculates the constraints implied by its existence. These conditions may be easily computed, using the standard G-structure formalism (as was done for eleven dimensional supergravity in [25, 26]), or, for maximal computational efficiency, the streamlined versions thereof given in [9], [27]. Then since ϵ is Killing, its isotropy group is promoted to the status of the preferred local structure group of the spacetime, and one has already constructed a spinorial basis decomposed into modules of the maximal compact factor of this structure group.

Second, one determines all the possible preferred local structure groups defined by arbitrary sets of additional Killing spinors, and the spaces of spinors they fix. This gives

a classification of the possible preferred local G-structures admitted by supersymmetric spacetimes in the supergravity, together with the number of supersymmetries consistent with each. The main objective of this paper is to perform this classification for sets of Killing spinors in eleven dimensions which contain at least one null Killing spinor but which are otherwise arbitrary.

Third, one computes the additional conditions on the intrinsic torsion and fluxes implied by the existence of an arbitrary additional Killing spinor η. Since η is of the form $\eta = Q\epsilon$, η is Killing if and only if

$$[\mathcal{D}_\mu, Q]\epsilon = 0, \tag{1}$$

where $\mathcal{D}_\mu$ is the supercovariant derivative. By imposing the defining projections satisfied by the fiducial Killing spinor ϵ, the spinor $[\mathcal{D}_\mu, Q]\epsilon$ may be written as a manifest sum of basis spinors, and by linear independence, the coefficient of each must vanish separately. The most convenient and concise way of expressing these conditions is to further exploit the existence of the G-structure defined by ϵ, and to decompose the spin connection and fluxes into modules of the maximal compact factor of the structure group. In this way the Killing spinor equation for η is converted into a set of purely bosonic equations for tensors of the maximal compact factor of the structure group defined by ϵ, which give additional constaints on the intrinsic torsion and fluxes. Once the general conditions for the existence of an arbitrary additional Killing spinor have been computed, the conditions implied by the existence of any particular set may be analysed in detail. In a forthcoming series of papers [35, 36, 37], we will compute the conditions implied by (1) for general Q and null ϵ in eleven dimensions.

In [28], some of the structure groups arising as subgroups of $SU(5)$ in eleven dimensions were classified. All such possible structure groups which can fix two Killing spinors, and some special cases of structure groups fixing $N > 2$, were given. The constraints on the bosonic fields associated with all $N = 2$ $SU(5)$ structures and some special cases of $SU(4)$ structures were computed. The description of spinors in [28] exploits the isomorphism between the space of Dirac spinors in eleven dimensional spacetime, and the space of $(0, p)$ forms, $p = 0, ..., 5$, in ten Riemannian dimensions. Though this language was not used explicitly in [27], the description of the spinors is manifestly equivalent; the authors of [28] construct the basis in spinor space by acting on a fiducial complex (timelike) spinor ρ with

$$R_{a_1...a_n}\Gamma^{a_1...a_n}, \tag{2}$$

where the $R_{a_1...a_n}$ given a basis for $(0, p)$ forms in ten Riemannian dimensions. By supressing ρ and the Gamma-matrices, the spinors in [28] are then treated as forms throughout. As was done in [27], this spinorial basis is then used to convert the Killing spinor equation into a set of purely bosonic equations.

As stated above, the objective of this work is to give the classification of all possible structure groups of supersymmetric eleven dimensional spacetimes which may be

embedded in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$, and to construct the spaces of spinors fixed by these groups. Since so many distinct cases are involved, the refined G-structure classification of supersymmetric spacetimes in eleven dimensions is a problem of great proportions. Here we will lay the algebraic groundwork for this problem, for all spacetimes with structure groups embedding in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. In section two we will construct a basis of spinors by acting on a fiducial null spinor with a subset of the Clifford algebra, and decompose the basis into modules of Spin(7). In section three we will use this basis to determine the possible structure groups, and the spaces of spinors they fix. In section four we discuss the structure groups which may be embedded both in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ and $SU(5)$ (so that the spacetimes with these structures admit both timelike and null Killing spinors), and determine the conditions on the spinors for this to occur. In particular, we will find some possible structure groups which may be embedded in $SU(5)$ which are not covered by [28]. Section five contains our conclusions. We also provide a sample of the sort of results that can be expected using our method [35]: the general local bosonic solution of the Killing spinor equation in eleven dimensions, admitting a Spin(7) structure.

2 The spinor basis

As discussed in the introduction, the most efficient way of analysing the Killing spinor equation with more than one supersymmetry is to act on a Killing spinor with a subset of the Clifford algebra, so that we may use (1). Since the constraints associated with the existence of a single null Killing spinor were computed in [26], we will adopt all their conventions so that their results may be readily incorporated into the refined classification. Before constructing the spinor basis, and to fix our notation, let us first summarise some useful material. In the spacetime basis

$$ds^2 = 2e^+e^- + e^ie^i + e^9e^9, \tag{3}$$

where $i = 1, ..., 8$, and where the eight dimensional manifold spanned by the e^i will be referred to as the base, a single null spinor ϵ may be fixed by the projections

$$\begin{aligned} \Gamma_{1234}\epsilon = \Gamma_{3456}\epsilon = \Gamma_{5678}\epsilon = \Gamma_{1357}\epsilon &= -\epsilon, \\ \Gamma^+\epsilon &= 0. \end{aligned} \tag{4}$$

The only non-zero bilinears which may be constructed from this spinor are the one form, two form and five form which are

$$\begin{aligned} K &= e^+, \\ \Omega &= e^+ \wedge e^9, \\ \Sigma &= e^+ \wedge \phi, \end{aligned} \tag{5}$$

where ϕ is the Spin(7) four form whose only non-zero components are given by

$$\begin{aligned} -\phi &= e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + e^{1357} \\ &- e^{1368} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457} + e^{2468}. \end{aligned} \tag{6}$$

This Spin(7) subgroup of Spin(8) is that which splits the positive chirality irrep $\mathbf{8}_+$ of Spin(8) into $\mathbf{8}_+ = \mathbf{1} + \mathbf{7}$ while leaving the negative chirality and vector irreps $\mathbf{8}_-$, $\mathbf{8}_v$ irreducible. Under Spin(7), the adjoint of Spin(8) splits as $\mathbf{28} = \mathbf{21} + \mathbf{7}$. For two forms, the projections may be written as

$$\begin{aligned} (P^{\mathbf{21}}A)_{ij} &= \frac{3}{4}(A_{ij} + \frac{1}{6}\phi_{ijkl}A^{kl}), \\ (P^{\mathbf{7}}A)_{ij} &= \frac{1}{4}(A_{ij} - \frac{1}{2}\phi_{ijkl}A^{kl}). \end{aligned} \tag{7}$$

We may choose a basis for the $\mathbf{7}$ to be

$$\begin{aligned} &J^1 = e^{18} + e^{27} - e^{36} - e^{45}, \qquad J^2 = e^{28} - e^{17} - e^{35} + e^{46}, \\ &J^3 = e^{38} + e^{47} + e^{16} + e^{25}, \qquad J^4 = e^{48} - e^{37} + e^{15} - e^{26} \\ &J^5 = e^{58} + e^{67} - e^{14} - e^{23}, \qquad J^6 = e^{68} - e^{57} - e^{13} + e^{24}, \\ &J^7 = e^{78} + e^{56} + e^{34} + e^{12}. \end{aligned} \tag{8}$$

They obey

$$\begin{aligned} J^A_{ik} J^{Bk}{}_j &= -\delta^{AB}\delta_{ij} + K^{AB}_{ij}, \\ J^{Aij} K^{BC}_{ij} &= 0, \\ K^{ABij} K^{CD}_{ij} &= 8(\delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC}), \end{aligned} \tag{9}$$

where $A, B = 1, ..., 7$ and the K^{AB}_{ij} are antisymmetric on A, B and furnish a basis for the $\mathbf{21}$. Note that

$$K^{AB}_{ij}\Gamma^{ij}\epsilon = 0, \tag{10}$$

$$J^A_{ij}\Gamma^{ij}\epsilon = 8\Gamma^{A8}\epsilon. \tag{11}$$

The most general element of the Lie algebra of Spin(1,10) which annihilates ϵ is thus obviously

$$f^{AB}K^{AB}_{ij}\Gamma^{ij} + \theta_i\Gamma^{+i} + q\Gamma^{+9} \tag{12}$$

which of course generates $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. Now we turn to the construction of the spinor basis. We define a basis of spinors by the projections (with no sum on i)

$$
\begin{aligned}
\Gamma_{1234}\epsilon_{(i)} &= -\alpha^1_{(i)}\epsilon_{(i)}, \\
\Gamma_{3456}\epsilon_{(i)} &= -\alpha^2_{(i)}\epsilon_{(i)}, \\
\Gamma_{5678}\epsilon_{(i)} &= -\alpha^3_{(i)}\epsilon_{(i)}, \\
\Gamma_{1357}\epsilon_{(i)} &= -\alpha^4_{(i)}\epsilon_{(i)}, \\
\Gamma^{\alpha^5_{(i)}}\epsilon_{(i)} &= 0,
\end{aligned} \tag{13}
$$

for the thirty-two possible combinations of $\alpha^{1,...,5}_{(i)} = \pm 1$. We want to construct this basis by acting on our fiducial spinor $\epsilon \equiv \epsilon_{(1)}$ (defined by $\alpha^{1,...,5}_{(1)} = +$) with a subset of the Clifford algebra. First consider the sixteen spinors with $\alpha^5_{(i)} = +$. We may split these spinors into two groups of eight, according to whether their chirality on the eight dimensional base is positive or negative (this chirality is given by $\alpha^1_{(i)}\alpha^3_{(i)}$). We may easily solve the projections (13) for the eight basis spinors with $\alpha^5_{(i)} = +$ and chirality $\alpha^1_{(i)}\alpha^3_{(i)} = +$ to find that they are given by

$$\epsilon, \;\; J^A_{ij}\Gamma^{ij}\epsilon, \tag{14}$$

where $A = 1, ..., 7$ and the splitting of the $\mathbf{8}_+$ of Spin(8) into the $\mathbf{1} + \mathbf{7}$ of Spin(7) is manifest. Next the eight basis spinors with $\alpha^5_{(i)} = +$, $\alpha^1_{(i)}\alpha^3_{(i)} = -$ are readily found to be

$$\Gamma^i\epsilon. \tag{15}$$

The eight basis spinors with $\alpha^5_{(i)} = -$, $\alpha^1_{(i)}\alpha^3_{(i)} = +$ are given by

$$\Gamma^-\epsilon, \;\; J^A_{ij}\Gamma^{-ij}\epsilon, \tag{16}$$

and the eight basis spinors with $\alpha^5_{(i)} = -$, $\alpha^1_{(i)}\alpha^3_{(i)} = -$ are

$$\Gamma^{-i}\epsilon. \tag{17}$$

Thus an arbitrary Majorana spinor η in eleven dimensions may be written as

$$\eta = (f + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij} + u_i\Gamma^i + g\Gamma^- + \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij} + v_i\Gamma^{-i})\epsilon, \tag{18}$$

for thirty-two real functions f, f^A, u_i, g, g^A, v_i, and the factors of 1/8 are introduced for future convenience. This equation is manifestly equivalent to the statement (in the language of [28]) that the space of Majorana spinors in eleven dimensions is isomorphic to the direct sum of the spaces of Spin(7) forms, defined on the eight dimensional base,

$$\Lambda^0_1 \oplus \Lambda^0_1 \oplus \Lambda^1_8 \oplus \Lambda^1_8 \oplus \Lambda^2_7 \oplus \Lambda^2_7. \tag{19}$$

3 The structure groups

In this section we will classify the possible structure groups arising as subgroups of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$, and the spaces of spinors they fix. This will give a complete classification of all G-structures which can arise for eleven dimensional spacetimes admitting at least one null Killing spinor, and will organise the enormously complicated problem of analysing the Killing spinor equation into many, more manageable, sub-problems. We will henceforth assume that the fiducial spinor ϵ is Killing, and we will determine the reduction of the structure group implied by the existence of all other sets of Killing spinors. An important point we will exploit throughout is our freedom to perform $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ transformations preserving the metric and leaving the fiducial spinor ϵ (and hence the bilinears (5)) invariant. This freedom will allow us to take some additional Killing spinors to be of a simple, "canonical" form; this point was also exploited in [28]. Let us also comment on our notation: throughout this section, the index A takes the values 1,...,7. B takes values 1,...,6, $C \in \{1, ..., 5\}$, $D \in \{1, ..., 4\}$, $E \in \{1, 2, 3\}$, $F \in \{1, 2\}$, $G \in \{6, 7\}$, $H \in \{5, 6, 7\}$, $I \in \{4, ..., 7\}$.

3.1 Additional Killing spinors of the form $(f + \frac{1}{8} f^A J^A_{ij} \Gamma^{ij})\epsilon$

In this subsection we will explore the reduction of the structure group implied by the existence of progressively more Killing spinors of the form $(f + \frac{1}{8} f^A J^A_{ij} \Gamma^{ij})\epsilon$. The structure groups which can arise by demanding Killing spinors of this form, together with the number of Killing spinors of this form they can fix, are summarised as follows:

G	N
$(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$	2
$(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$	3
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$	4
$(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$	5
$(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$	6
Chiral $\mathbb{R}^9$	7,8

By something of an abuse of language, we take a "chiral $\mathbb{R}^9$ structure" to refer to a G-structure defined by a set of Killing spinors with common isotropy group $\mathbb{R}^9$ which share the same chirality on the eight dimensional base. Let us now consider each case in more detail.

$(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ **structure, N=2.** Suppose we demand the existence of a second linearly independent Killing spinor of the form

$$\eta_{(1)} = (f_{(1)} + \frac{1}{8} f^A_{(1)} J^A_{ij} \Gamma^{ij})\epsilon. \qquad (20)$$

We will now exploit the fact that Spin(7) acts transitively on the unit sphere in the **7**. We may perform a Spin(7) transformation, preserving the metric, ϵ and the bilinears, to write $\eta_{(1)}$ as

$$\eta_{(1)} = (f_{(1)} + \frac{1}{8} g_{(1)} J^7_{ij} \Gamma^{ij})\epsilon, \tag{21}$$

where $(g_{(1)})^2 = f^A_{(1)} f^A_{(1)}$. Since

$$\frac{Spin(d+1)}{Spin(d)} \cong S^d, \tag{22}$$

the stabiliser of $\eta_{(1)}$ in Spin(7) is Spin(6)$\cong SU(4)$, and the existence of this Killing spinor implies that the structure group is reduced to $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$. This may be verified directly by applying the general Lie algebra element (12) to $\eta_{(1)}$. Using (9) and (10), it is straightforward to verify that $\eta_{(1)}$ is annihilated if and only if $f^{7A} = 0$ in (12). The common isotropy algebra of ϵ and $\eta_{(1)}$ is thus that of $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$, and this structure group fixes precisely two spinors. $\eta_{(1)}$ is linearly independent of ϵ if $g_{(1)} \neq 0$, but the most generic additional spinor consistent with an $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure has $f_{(1)} \neq 0$.

$(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ **structure, N=3.** Suppose we demand, in addition to ϵ and $\eta_{(1)}$, the existence of a third linearly independent Killing spinor of the form

$$\eta_{(2)} = \eta_{SU(4)\ltimes} + \frac{1}{8} f^B_{(2)} J^B_{ij} \Gamma^{ij} \epsilon, \tag{23}$$

where $\eta_{SU(4)\ltimes}$ is an arbitrary spinor fixed by the $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ of the previous paragraph, and $1/8 f^B_{(2)} J^B_{ij} \Gamma^{ij} \epsilon$, $B = 1, ..., 6$ is an arbitrary non-zero spinor in the **6** of Spin(6). Using the fact that Spin(6) acts transitively on the unit sphere its **6** with stabiliser Spin(5) $\cong Sp(2)$, we may write $\eta_{(2)}$ as

$$\eta_{(2)} = \eta_{SU(4)\ltimes} + \frac{1}{8} g_{(2)} J^6_{ij} \Gamma^{ij} \epsilon, \tag{24}$$

and deduce that the common isotropy group of ϵ, $\eta_{(1)}$ and $\eta_{(2)}$ is $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$. Together with the other two Killing spinors, $\eta_{(2)}$ is annihilated if and only if in addition $f^{6A} = 0$ in (12).

$((SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$ **structure, N=4.** Suppose we demand the existence of a fourth linearly independent Killing spinor of the form

$$\eta_{(3)} = \eta_{Sp(2)\ltimes} + \frac{1}{8} f^C_{(3)} J^C_{ij} \Gamma^{ij} \epsilon, \tag{25}$$

where $\eta_{Sp(2)\ltimes}$ is an arbitrary spinor fixed by the $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ of the previous paragraph, and $1/8 f^C_{(3)} J^C_{ij} \Gamma^{ij}\epsilon$, $C = 1, ..., 5$ is a non-zero spinor in the **5** of Spin(5). Since Spin(5) acts transitively on the unit sphere in its **5** with stabiliser Spin(4) $\cong$ $SU(2) \times SU(2)$, we may write $\eta_{(3)}$ as

$$\eta_{(3)} = \eta_{Sp(2)\ltimes} + \frac{1}{8} g_{(3)} J^5_{ij} \Gamma^{ij}\epsilon, \tag{26}$$

and deduce that the existence of this Killing spinor in addition to the previous three implies that the structure group is reduced to $((SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$. Together with the other three Killing spinors, $\eta_{(3)}$ is annihilated if and only if in addition $f^{5A} = 0$ in (12).

$(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ **structure, N=5.** Suppose we demand the existence of a fifth linearly independent Killing spinor of the form

$$\eta_{(4)} = \eta_{SU(2)\times SU(2)\ltimes} + \frac{1}{8} f^D_{(4)} J^D_{ij} \Gamma^{ij}\epsilon, \tag{27}$$

where $\eta_{SU(2)\times SU(2)\ltimes}$ is an arbitrary spinor fixed by the $((SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$ of the previous paragraph and $1/8 f^D_{(4)} J^D_{ij} \Gamma^{ij}\epsilon$, $D = 1, ..., 4$ is a non-zero spinor in the **4** of Spin(4). Since Spin(4) acts transitively on the unit sphere in its **4** with stabiliser Spin(3) $\cong SU(2)$, we may write $\eta_{(4)}$ as

$$\eta_{(4)} = \eta_{SU(2)\times SU(2)\ltimes} + \frac{1}{8} g_{(4)} J^4_{ij} \Gamma^{ij}\epsilon, \tag{28}$$

and deduce that the existence of this Killing spinor in addition to the previous four implies that the structure group is reduced to $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$. Together with the other four Killing spinors, $\eta_{(4)}$ is annihilated if and only if in addition $f^{4A} = 0$ in (12). Note that the $SU(2)$ factor in this subgroup of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ acts nontrivially on all eight dimensions of the base. It is certainly not true that a holonomy group that is strictly of this form (and not some subgroup) can arise, since an $SU(2)$ holonomy acting nontrivially on eight dimensions is not on Berger's list. In the abscence of flux (so the intrinsic torsion of the G-structure vanishes), the existence of the Killing spinor $\eta_{(4)}$ in addition to the other four would identically imply the existence of more Killing spinors, which would reduce the holonomy group to $\mathbb{R}^9$, with $N = 16$. When the flux is non-zero, the intrinsic torsion of the G-structure is non-zero, and we are unaware of any theorems forbidding an $SU(2)$ structure (as opposed to holonomy) in eight dimensions. However, what we are doing in this paper is purely algebraic, and it may well be that a direct analysis of the Killing spinor equation will reveal that the existence of the Killing spinor $\eta_{(4)}$ in addition to the other four will always identically imply the existence of more Killing spinors, so that the structure group would be further reduced, and strictly $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures would not arise.

$(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, N=6. Suppose we demand the existence of a sixth linearly independent Killing spinor of the form

$$\eta_{(5)} = \eta_{SU(2)\ltimes} + \frac{1}{8} f^E_{(5)} J^E_{ij} \Gamma^{ij} \epsilon, \tag{29}$$

where $\eta_{SU(2)\ltimes}$ is an arbitrary spinor fixed by the $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ of the previous paragraph, and $\frac{1}{8} f^E_{(5)} J^E_{ij} \Gamma^{ij} \epsilon$, $E = 1, 2, 3$ is a non-zero spinor in the **3** of Spin(3). Since Spin(3) acts transitively on the unit sphere in its **3** with stabiliser Spin(2) $\cong U(1)$, we may write $\eta_{(5)}$ as

$$\eta_{(5)} = \eta_{SU(2)\ltimes} + \frac{1}{8} g_{(5)} J^3_{ij} \Gamma^{ij} \epsilon, \tag{30}$$

and deduce that the existence of this Killing spinor in addition to the previous five implies that the structure group is reduced to $(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$. Together with the other five Killing spinors, $\eta_{(5)}$ is annihilated if and only if in addition $f^{3A} = 0$ in (12). The same remarks as those made regarding the existence or otherwise of the $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure apply to this case.

Chiral $\mathbb{R}^9$ structure, N=7,8. Suppose we demand the existence of a seventh linearly independent Killing spinor of the form

$$\eta_{(6)} = \eta_{U(1)\ltimes} + \frac{1}{8} f^F_{(6)} J^F_{ij} \Gamma^{ij} \epsilon, \tag{31}$$

where $\eta_{U(1)\ltimes}$ is an arbitrary spinor fixed by the $(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$ of the previous paragraph, and $1/8 f^F_{(6)} J^F_{ij} \Gamma^{ij} \epsilon$, $F = 1, 2$ is a non-zero spinor in the **2** of Spin(2). Since Spin(2) acts transitively on the unit sphere in its **2** with stabiliser the identity, we may write $\eta_{(6)}$ as

$$\eta_{(6)} = \eta_{U(1)\ltimes} + \frac{1}{8} g_{(6)} J^2_{ij} \Gamma^{ij} \epsilon, \tag{32}$$

and deduce that the existence of this Killing spinor in addition to the previous six implies that the structure group is reduced to $\mathbb{R}^9$. Together with the other six Killing spinors, $\eta_{(6)}$ is annihilated if and only if $f^{AB} = 0$ in (12). We have used the last of the Spin(7) freedom to fix $\eta_{(6)}$ in the form (32). We can have one further linearly independent Killing spinor with a chiral $\mathbb{R}^9$ structure. This is $\eta_{(7)} = (f_{(7)} + \frac{1}{8} f^A_{(7)} J^A_{ij} \Gamma^{ij})\epsilon$, $f^1_{(7)} \neq 0$. We have no freedom left to fix any of the components of this spinor while preserving the form of the first seven, so generically all eight functions in $\eta_{(7)}$ are nonzero.

3.2 Additional Killing spinors of the form $(f + \frac{1}{8} f^A J^A_{ij} \Gamma^{ij} + u_i \Gamma^i)\epsilon$

In this subsection we will examine the reduction of the structure group implied by incorporating progressively more Killing spinors of the more generic form $(f + \frac{1}{8} f^A J^A_{ij} \Gamma^{ij} +$

$u_i\Gamma^i)\epsilon$. The structure groups which can arise (distinct from those of the previous subsection, which are obtained with $u_i = 0$), and the number of Killing spinors they can fix, are summarised as follows:

G	N
$(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$	2
$(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$	2,3,4
$(SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}^5$	3,4,...,8
$\mathbb{R}^9$	3,4,...,16

Let us discuss each case in more detail. We point out that we reserve the notation σ for Killing spinors which are completely generic among spinors of the form discussed in this subsection. $\sigma_{(i)}$ denotes the ith such Killing spinor.

$(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$ structure, N=2,3,4. Suppose we demand the existence of a second independent Killing spinor, of the most generic form discussed in this subsection,

$$\sigma_{(1)} = (f_{(1)} + \frac{1}{8} f^A_{(1)} J^A_{ij} \Gamma^{ij} + u_{(1)i}\Gamma^i)\epsilon. \tag{33}$$

where $f^A_{(1)},\ u_{(1)i} \neq 0$. We may use the fact that Spin(7) acts transitively on the unit sphere in its $\mathbf{8}_-$, and that

$$\frac{Spin(7)}{G_2} \cong S^7, \tag{34}$$

to deduce that we may take

$$\sigma_{(1)} = (f_{(1)} + \frac{1}{8} f^A_{(1)} J^A_{ij} \Gamma^{ij} + u_{(1)}\Gamma^8)\epsilon, \tag{35}$$

where $(u_{(1)})^2 = u_{(1)i}u^i_{(1)}$, and we still have the freedom to perform G_2 transformations preserving $u_{(1)i} = u_{(1)}\delta_{i8}$. Under G_2, the $\mathbf{8}_-$ of Spin(7) decomposes as $\mathbf{8}_- = \mathbf{1}+\mathbf{7}_{G_2}$. The $\mathbf{7}$ of Spin(7) is left irreducible under G_2. We now use the fact that G_2 acts transitively on the unit sphere in its $\mathbf{7}$, and that

$$\frac{G_2}{SU(3)} \cong S^6, \tag{36}$$

to deduce that we may take

$$\sigma_{(1)} = (f_{(1)} + \frac{1}{8} g_{(1)} J^7_{ij} \Gamma^{ij} + u_{(1)}\Gamma^8)\epsilon, \tag{37}$$

and the existence of $\sigma_{(1)}$ implies that the structure group is reduced to $(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$, where the $SU(3)$ acts nontrivially in the 123456 directions. The $\mathbf{7}$ of G_2 decomposes

under $SU(3)$ as $\mathbf{7} = \mathbf{1} + \mathbf{3} + \overline{\mathbf{3}}$. Using (11), we note that $\sigma_{(1)}$ is annihilated by (12) if and only if $f^{AB}K^{AB}_{i8} = f^{AB}K^{AB}_{i7} = 0$. These are thirteen independent conditions, agreeing with the fact that the Lie algebra of $SU(3)$ is eight dimensional. Finally, we note that this structure group fixes the four basis spinors ϵ, $J^7_{ij}\Gamma^{ij}\epsilon$, $\Gamma^7\epsilon$ and $\Gamma^8\epsilon$. It is thus consistent with the existence of 2, 3 or 4 Killing spinors.

$(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$ **structure, N=2.** Suppose that we demand, instead of the most generic (of the form discussed in this subsection) second Killing spinor $\sigma_{(1)}$ of the previous paragraph, that $f^A = 0$ in (33). In this case, we may take the second Killing spinor to be

$$\alpha = (f_{(1)} + u_{(1)}\Gamma^8)\epsilon, \tag{38}$$

and the existence of this Killing spinor implies that the structure group is reduced to $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$; the G_2 acts nontrivially on the 1234567 directions. It is straightforward to verify that in this case α is annihilated by (12) if and only if $f^{AB}K^{AB}_{i8} = 0$. These are seven independent conditions, which agrees with the fact that the Lie algebra of G_2 is fourteen dimensional. This structure group stabilises two basis spinors, and is thus consistent with $N = 2$.

$\mathbb{R}^9$ **structure, N=3,4,...,16.** Next let us suppose that in addition to the generic (of the form discussed in this subsection) pair of Killing spinors ϵ, $\sigma_{(1)}$, we demand the existence of a generic third Killing spinor

$$\sigma_{(2)} = \eta_{SU(3)\ltimes} + (f_{(2)B}\Gamma^{B8} + u_{(2)B}\Gamma^B)\epsilon, \tag{39}$$

where $\eta_{SU(3)\ltimes}$ is an arbitrary spinor fixed by the $(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$ given above, $f_{(2)B}$, $u_{(2)B}$, $B = 1, ..., 6$ are arbitrary generic non-zero one forms in the $\mathbf{3}+\overline{\mathbf{3}}$ of $SU(3)$ and we have reexpressed $J^A_{ij}\Gamma^{ij}\epsilon$ using (11). We may exploit the fact that $SU(3)$ acts transitively on the unit sphere in its $\mathbf{3} + \overline{\mathbf{3}}$, and that

$$\frac{SU(3)}{SU(2)} \cong S^5 \tag{40}$$

to set

$$\sigma_{(2)} = \eta_{SU(3)\ltimes} + (f_{(2)}\Gamma^{68} + u_{(2)6}\Gamma^6 + u_{(2)5}\Gamma^5 + u_{(2)D}\Gamma^D)\epsilon, \tag{41}$$

where $D = 1, ...4$ and $u_{(2)D}$ is a non-zero one form in the $\mathbf{2} + \overline{\mathbf{2}}$ of $SU(2)$. We still have the freedom to perform $SU(2)$ transformations in the 1234 directions preserving $f_{(2)B} = \delta_{6B}f_{(2)}$, $B = 1, ..., 6$. We may use the fact that $SU(2)$ acts transitively on the unit sphere in its $\mathbf{2} + \overline{\mathbf{2}}$, and

$$SU(2) \cong S^3, \tag{42}$$

to write $\sigma_{(2)}$ as

$$\sigma_{(2)} = \eta_{SU(3)\ltimes} + (f_{(2)}\Gamma^{68} + u_{(2)6}\Gamma^6 + u_{(2)5}\Gamma^5 + u_{(2)4}\Gamma^4)\epsilon. \qquad (43)$$

and deduce that the existence of the Killing spinors ϵ, $\sigma_{(1)}$, and $\sigma_{(2)}$ implies that the structure group is reduced to $\mathbb{R}^9$. These Killing spinors are annihilated by (12) if and only if $f^{AB}K^{AB}_{ij} = 0$. An $\mathbb{R}^9$ structure group fixes the sixteen basis spinors ϵ, $J^A_{ij}\Gamma^{ij}\epsilon$, $\Gamma^i\epsilon$, and is thus consistent with $N = 3, ..., 16$.

$(SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}^5)$ **structure, N=3,4,...,8.** Let us suppose that instead of the generic third Killing spinor $\sigma_{(2)}$ (of the form discussed in this subsection) there exists a third Killing spinor with $u_{(2)D} = 0$, $D = 1, ..., 4$ in (41). Then demanding that this non-generic third Killing spinor is annihilated by (12) in addition to the generic first two implies that in addition to the conditions $f^{AB}K^{AB}_{i8} = f^{AB}K^{AB}_{i7} = 0$, we must have the five further conditions $f^{AB}K^{AB}_{i6} = 0$. Note that these conditions identically imply that $f^{AB}K^{AB}_{i5} = 0$. The existence of a third Killing spinor of this non-generic form thus implies that the structure group is reduced to $(SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}^5$, where the $SU(2)$ acts nontrivially in the 1234 directions. This structure group stabilises the eight basis spinors ϵ, $J^H_{ij}\Gamma^{ij}\epsilon$, $\Gamma^H\epsilon$, $H = 5, 6, 7$, and is thus consistent with $N = 3, ..., 8$.

3.3 Additional Killing spinors of the form $(f + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij} + g\Gamma^- + \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij})\epsilon$

In this subsection we will examine the reduction of the structure group implied by incorporating progressively more Killing spinors of the form

$$(f + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij} + g\Gamma^- + \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij})\epsilon. \qquad (44)$$

The structure groups which can arise (distinct from those of the subsection before last, which are obtained with $g = g^A = 0$), and the number of Killing spinors of this form they can fix, are summarised as follows:

G	N
Spin(7)	2
$SU(4)$	2,3,4
$Sp(2)$	2,3,...,6
$SU(2) \times SU(2)$	3,4,...,8
Chiral $SU(2)$	3,4,...,10
$U(1)$	4,5,...,12
Chiral Identity	4,5...16

Let us consider each case in more detail. We reserve the notation ξ for Killing spinors which are completely generic among those discussed in this subsection. $\xi_{(i)}$ denotes the ith such spinor. Note that for any configuration admitting multiple Killing spinors of the form (44), with at least one of $g, g^A \neq 0$, we have the freedom to act with the $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ element

$$1 + q\Gamma^{+9}, \tag{45}$$

so that by an appropriate choice of q one of the Killing spinors may be chosen such that one of the $f, f^A = 0$, since ϵ satisfies

$$\Gamma^{+-}\epsilon = \Gamma^9\epsilon = \epsilon. \tag{46}$$

$Spin(7)$ **structure, N=2.** Suppose we demand, in addition to ϵ, the existence of the non-generic second Killing spinor

$$\alpha = (f + g\Gamma^-)\epsilon. \tag{47}$$

The existence of this Killing spinor implies that the structure group is reduced to Spin(7). α is annihilated by (12) if and only if $\theta_i = q = 0$.

$SU(4)$ **structure, N=2,3,4.** Instead of α, suppose we demand the existence of the non-generic second Killing spinor

$$\beta = \eta_{Spin(7)} + \frac{1}{8} f^A J^A_{ij}(g\Gamma^{ij} + h\Gamma^{-ij})\epsilon, \tag{48}$$

where $\eta_{Spin(7)}$ is an arbitrary spinor fixed by the Spin(7) of the previous paragraph, $f^A \neq 0$ and $\Gamma^+\beta \neq 0$. Since a Spin(7) transformation on the spinor irreps commutes with Γ^-, we can perform a Spin(7) transformation to set

$$\beta = \eta_{Spin(7)} + \frac{1}{8} f J^7_{ij}(g\Gamma^{ij} + h\Gamma^{-ij})\epsilon, \tag{49}$$

and this is annihilated by (12) if and only if $f^{7A} = \theta_i = q = 0$. The stabiliser of ϵ and β is thus $SU(4)$, and since this group fixes the four basis spinors ϵ, $\Gamma^-\epsilon$, $J^7_{ij}\Gamma^{ij}\epsilon$, $J^7_{ij}\Gamma^{-ij}\epsilon$, it is consistent with $N = 2, 3, 4$.

$Sp(2)$ **structure, N=2,3,...,6.** Instead of α or β, suppose we demand the existence of the most generic (of the form discussed in this subsection) second Killing spinor $\xi_{(1)}$,

$$\xi_{(1)} = \eta_{Spin(7)} + (\frac{1}{8} f^A J^A_{ij}\Gamma^{ij} + \frac{1}{8} g^A J^A_{ij}\Gamma^{-ij})\epsilon, \tag{50}$$

where f^A, g^A are non-zero and not linearly dependent. Then from subsection 3.1, we may set

$$\xi_{(1)} = \eta_{SU(4)} + \frac{1}{8} f_{(1)} J^6_{ij} \Gamma^{ij} \epsilon, \tag{51}$$

where $\eta_{SU(4)}$ is an arbitrary spinor fixed by the $SU(4)$ of the previous paragraph. It is easy to verify that the stabiliser of ϵ and $\xi_{(1)}$ is $Sp(2)$. Since this group stabilises the six basis spinors ϵ, $\Gamma^- \epsilon$, $J^7_{ij}\Gamma^{ij}\epsilon$, $J^7_{ij}\Gamma^{-ij}\epsilon$, $J^6_{ij}\Gamma^{ij}\epsilon$, $J^6_{ij}\Gamma^{-ij}\epsilon$, it is consistent with $N = 2, 3, ..., 6$.

$SU(2) \times SU(2)$ **structure, N=3,4,...,8.** In addition to the two most generic (of the form discussed in this subsection) Killing spinors ϵ, $\xi_{(1)}$, suppose we demand the existence of the non-generic third Killing spinor

$$\gamma = \eta_{Sp(2)} + \frac{1}{8} f^C J^C_{ij} (g\Gamma^{ij} + h\Gamma^{-ij})\epsilon, \tag{52}$$

where $\eta_{Sp(2)}$ is an arbitrary spinor fixed by the $Sp(2)$ of the previous paragraph, $C = 1, ..., 5$, $f^C \neq 0$ and $\Gamma^+ \gamma \neq 0$. We may set

$$\gamma = \eta_{Sp(2)} + \frac{1}{8} f J^5_{ij} (g\Gamma^{ij} + h\Gamma^{-ij})\epsilon, \tag{53}$$

and deduce that the existence of this third Killing spinor γ implies that the structure group is reduced to $SU(2) \times SU(2)$. This group fixes eight of the basis spinors, so is consistent with $N = 3, ..., 8$.

Chiral $SU(2)$**, N=3,4,...,10.** Intead of γ, suppose we demand, in addition to ϵ and $\xi_{(1)}$, the existence of the most generic (of the form of this subsection) third Killing spinor $\xi_{(2)}$. We may take this to be

$$\xi_{(2)} = \eta_{SU(2)\times SU(2)} + \frac{1}{8} f_{(2)} J^4_{ij} \Gamma^{ij} \epsilon, \tag{54}$$

where $\eta_{SU(2)\times SU(2)}$ is an arbitrary spinor fixed by the $SU(2) \times SU(2)$ of the previous paragraph. In addition to the other two, this Killing spinor is fixed by an $SU(2)$ acting non-trivially in eight dimensions. We refer to this structure as a chiral $SU(2)$ structure, and the same remarks made for the case of the $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure apply in this case. Since this $SU(2)$ fixes ten basis spinors, it is consistent with $N = 3, ..., 10$.

$U(1)$ **structure, N=4,...,12** Suppose we demand, in addition to the Killing spinors ϵ, $\xi_{(1)}$ and $\xi_{(2)}$, the fourth non-generic Killing spinor

$$\delta = \eta_{CSU(2)} + \frac{1}{8} f^E J^E_{ij} (g\Gamma^{ij} + h\Gamma^{-ij})\epsilon, \tag{55}$$

where $\eta_{CSU(2)}$ is an arbitrary spinor fixed by the chiral $SU(2)$ of the previous paragraph, $E=1,2,3,\ f^E \neq 0$ and $\Gamma^+\delta \neq 0$. We may set

$$\delta = \eta_{CSU(2)} + \frac{1}{8} f J^3_{ij}(g\Gamma^{ij} + h\Gamma^{-ij})\epsilon, \tag{56}$$

and in addition to the other three Killing spinors, this is fixed by $U(1)$. Since this group fixes twelve basis spinors, it is consistent with $N=4,...,12$. The same remarks made for $(SU(2)\ltimes\mathbb{R}^8)\times\mathbb{R}$ apply in this case.

Chiral Identity structure, N=4,...,16 Instead of δ, suppose we demand, in addition to ϵ, $\xi_{(1)}$ and $\xi_{(2)}$, the existence of the most generic (of the form discussed in this subsection) fourth Killing spinor. We may take this to be of the form

$$\xi_{(3)} = \eta_{U(1)} + \frac{1}{8} f_{(3)} J^2_{ij}\Gamma^{ij}\epsilon, \tag{57}$$

where $\eta_{U(1)}$ is an arbitrary spinor fixed by the $U(1)$ of the previous paragraph. The existence of $\xi_{(3)}$ in addition to the other three Klling spinors reduces the structure group to the identity. An identity structure is thus consistent with 4,...,16 Killing spinors of the same chirality on the base.

3.4 Additional Killing spinors of generic form

Finally let us discuss the case where the additional Killing spinors are assumed to be of the most generic form,

$$(f + \frac{1}{8} f^A J^A_{ij}\Gamma^{ij} + u_i\Gamma^i + g\Gamma^- + \frac{1}{8} g^A J^A_{ij}\Gamma^{-ij} + v_i\Gamma^{-i})\epsilon. \tag{58}$$

The new structure groups we find in this case, together with the number of Killing spinors they can fix, are as follows:

G	N
G_2	2,3,4
$SU(3)$	2,3,...,8
$SU(2)$	2,3,...,16
Identity	3,4,...32

$SU(2)$ **structure, N=2,3,...,32.** Before we obtain the structure group defined by the most generic second Killing spinor, consider first a second Killing spinor of the form (58), but with $v_i = 0$, and at least one of $g, g^A \neq 0$. We may act on this Killing spinor with the $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R})$ element

$$1 + \theta_i\Gamma^{+i}, \tag{59}$$

where

$$\theta_i = \frac{1}{(g^2 + g^A g^A)}(g u_i + g^A J^A_{ij} u^j). \tag{60}$$

Given the projections (46) satisfied by ϵ, together with

$$\begin{aligned} \Gamma_{ijk}\epsilon &= -\phi_{ijkl}\Gamma^l \epsilon, \\ \phi_{ijkl} J^{Akl} &= -6 J^A_{ij}, \end{aligned} \tag{61}$$

we see that we may set $u_i = 0$, and thus take the Killing spinor to be of the form discussed in the previous subsection. So now consider the case of the most generic second Killing spinor, given by (58) with all thirty-two functions (and in particular, the v_i) non-zero. We can act on the $\mathbf{8}_-$ spanned by the $\Gamma^{-i}\epsilon$ with Spin(7) to set these components to be of the form

$$v\Gamma^{-8}\epsilon. \tag{62}$$

We have the freedom to perform $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$ transformations preserving this form. By acting with

$$1 + \theta_A \Gamma^{+A} + \theta_8 \Gamma^{+8} + g\Gamma^{+9}, \tag{63}$$

we may set $f = f^A = u_8 = 0$, and we still have the freedom to perform G_2 transformations. We exploit this to set the $\mathbf{7}_{G_2}$ spanned by the $\Gamma^A\epsilon$ to be

$$u\Gamma^7\epsilon, \tag{64}$$

and we still have the freedom to perform $SU(3)$ transformations preserving (62), (64), and $f = f^A = 0$. We exploit this to set the components in the $\mathbf{7} = \mathbf{1} + (\mathbf{3} + \bar{\mathbf{3}})$ spanned by the $J^A_{ij}\Gamma^{-ij}$ to be of the form

$$\frac{1}{8} g^G J^G_{ij} \Gamma^{-ij}\epsilon, \tag{65}$$

where $G = 6, 7$. We have now completely fixed the most generic second Killing spinor; it takes the form

$$\lambda_{(1)} = (u\Gamma^7 + g\Gamma^- + \frac{1}{8} g^G J^G_{ij}\Gamma^{-ij} + v\Gamma^{-8})\epsilon, \tag{66}$$

where $G = 6, 7$. Thus, the common stabiliser of the most generic $\lambda_{(1)}$ and ϵ is $SU(2)$, acting on the 1234 directions. Such a group fixes sixteen basis spinors, and so is consistent with $N = 2, 3, ..., 16$.

$SU(3)$ **structure, N=2,3,...,8.** Suppose that in (66) $g^6 = 0$. Then $\lambda_{(1)}$ and ϵ are stabilised by an $SU(3)$ acting in the 123456 directions. This structure group stabilises eight basis spinors, and is thus consistent with $N = 2, 3, ..., 8$.

G_2 structure, N=2,3,4. Suppose that in (66), $g^6 = g^7 = u = 0$. Then $\lambda_{(1)}$ and ϵ are stabilised by a G_2 acting in the 1234567 directions. This structure group stabilises four basis spinors, and is thus consistent with $N = 2, 3, 4$.

Identity structure, $N = 3, 4, ..., 32$ Incorporating a third generic Killing spinor in addition to ϵ and the generic $\lambda(1)$ reduces the structure group to the identity. An identity structure is thus consistent with $N = 3, 4, ..., 32$. This completes the classification of subgroups of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ which can arise as structure groups in eleven dimensions.

4 Structures which embed both in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ and in $SU(5)$

In the previous section we classified all structure groups which can arise as subgroups of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ - that is, all groups which can arise as structure groups of eleven dimensional spacetimes admitting at least one null Killing spinor. When the analysis begun in [28] is completed, the analagous question will be answered for all eleven dimensional spacetimes admitting at least one timelike Killing spinor, where the structure group is a subgroup of $SU(5)$. There can clearly be a redundancy in the classification for structure groups that are subgroups both of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ and of $SU(5)$; some (but not all) spacetimes with these structure groups can admit both timelike and null Killing spinors. To lift this redundancy, we will make the following definitions:

A supersymmetric spacetime is said to admit a timelike G-structure if it admits only timelike Killing spinors.

A supersymmetric spacetime is said to admit a mixed G-structure if it admits both timelike and null Killing spinors.

A supersymmetric spacetime is said to admit a null G-structure if it admits only null Killing spinors.

Spacetimes admitting timelike G-structures may be classified using the framework of [25], [28], and spacetimes admitting null G-structures can be classified using the framework of [26] and this paper. Spacetimes admitting mixed G-structures can be classified using either. We now want to determine which of the G-structures of the previous section are null and which are mixed. Those G-structures for which G is not a subgroup of $SU(5)$ are trivially null. A complete list of such groups is given by $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ itself, all the groups of subsection 3.1, all the groups of subsection 3.2, Spin(7), and G_2.

The remaining groups of subsections 3.3 and 3.4 are subgroups of both $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ and $SU(5)$, so we must analyse these cases in more detail.

Let us thus assume that ϵ and χ are Killing, where χ is assumed to be of the general form

$$\chi = (f + \frac{1}{8} f^A J^A_{ij} \Gamma^{ij} + u_i \Gamma^i + g\Gamma^- + \frac{1}{8} g^A J^A_{ij} \Gamma^{-ij} + v_i \Gamma^{-i})\epsilon. \tag{67}$$

The fiducial Killing spinor ϵ is null. If χ is null, it may still be that the linear combination $\chi + \rho\epsilon$ is timelike, for some $\rho \in \mathbb{R}$. Thus the G-structure defined by ϵ and χ is null if and only if the Killing spinor $\chi + \rho\epsilon$ is null for all $\rho \in \mathbb{R}$. Let us compute the one form V with components

$$V_\mu = (\overline{\chi + \rho\epsilon})\Gamma_\mu(\chi + \rho\epsilon), \tag{68}$$

by setting $f \to f + \rho$ in (67). We find

$$\begin{aligned} V &= ((f+\rho)^2 + u_i u^i + f^A f^A)e^+ - 2(g^2 + v_i v^i + g^A g^A)e^- \\ &+ 2(v_i u^i - (f+\rho)g - f^A g^A)e^9 + 2(g^A u^i J^A_{ij} + f^A v^i J^A_{ij} - (f+\rho)v_j - g u_j)e^j \end{aligned} \tag{69}$$

Now, using the identity

$$u^i J^A_{ij} v^j u^k J^A_{kl} v^l = u_i u^i v_j v^j - (u_i v^i)^2, \tag{70}$$

we find that

$$V_\mu V^\mu = -[(f+\rho)g^A - g f^A + u^i J^A_{ij} v^j][(f+\rho)g^A - g f^A + u^k J^A_{kl} v^l] - 2f^{[A} g^{B]} f^A g^B. \tag{71}$$

Thus, $V_\mu V^\mu = 0$ for all $\rho \in \mathbb{R}$ if and only if

$$g^A = 0, \tag{72}$$

$$g f^A = u^i J^A_{ij} v^j, \tag{73}$$

and these are the necessary and sufficient conditions on the spinor χ for the G-structure defined by ϵ, χ to be null. A G-structure which is defined by more than two Killing spinors is null if and only if these conditions hold for all linear combinations of the Killing spinors with constant coefficients.

The G-structures with structure groups $SU(4)$, $Sp(2)$, $SU(2) \times SU(2)$, Chiral $SU(2)$, $U(1)$ or Chiral Identity are never null, since they always admit Killing spinors with $g^A \neq 0$, or $g f^A \neq 0$, $u^i = v^i = 0$. Thus, if they embed in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$, these structure groups may always be embedded in $SU(5)$ as well, and these G-structures are thus mixed. It will be most efficient to analyse these cases using the timelike basis of spinors of [28], since one can also have timelike G-structures for these groups.

Finally we note that although $SU(3)$, $SU(2)$ and the Identity are subgroups of both $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ and $SU(5)$, it is possible to have null $SU(3)$, $SU(2)$ or Identity

structures if (72), (73) hold for all linear combinations of the associated Killing spinors with constant coefficients. One may also have mixed or timelike G-structures with these structure groups.

In summary, the G-structures which are always null in eleven dimensions have structure groups

G
$(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$
$(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$
$(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$
$(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$
$(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$
Chiral $\mathbb{R}^9$
$(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$
$(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$
$(SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}^5$
$\mathbb{R}^9$
$Spin(7)$
G_2

The G-structures which, although their structure groups may be embedded in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$, are always mixed or timelike, are as follows:

G
$SU(4)$
$Sp(2)$
$SU(2) \times SU(2)$
Chiral $SU(2)$
$U(1)$
Chiral Identity

The G-structures which can be null if and only if (72), (73) hold for all linear combinations of the associated Killing spinors with constant coefficients, and which can otherwise be mixed or timelike, have structure groups

G
$SU(3)$
$SU(2)$
Identity

5 Conclusions

In this work we have classified all structure groups which can arise as subgroups of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ in eleven dimensions, and we have constructed the spaces of spinors they fix. We have introduced the notions of timelike, mixed and null G-structures for supersymmetric spacetimes. We have further classified the structure subgroups of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ according to these definitions.

The objective of the refined G-structure classification scheme in this context is to categorize the general local bosonic solution of the Killing spinor equation of eleven dimensional supergravity. We believe that this is a viable objective. The idea is ultimately to cover all possible multi-spinor ansätze for the Killing spinor equation, in full generality. In this work, the different types of ansatz assuming at least one null Killing spinor have been classified, with the common isotropy group of the spinors providing the organising principle. Since the spinors are assumed to be Killing, their common isotropy group is elevated to the status of the structure group of the supersymmetric spacetimes admitting those Killing spinors.

Now that the possible types of multi-spinor ansätze with at least one null spinor have been classified, it remains to perform a direct analysis of the Killing spinor equation. As discussed in the introduction, this will proceed by imposing

$$[\mathcal{D}_\mu, Q]\epsilon = 0, \tag{74}$$

for the most general choice of N distinct Qs for every N for every structure group discussed in this paper and every structure group arising in the timelike case (though in certain cases, we expect that the analysis of the Killing spinor equation will reveal that the existence of a given set of Killing spinors will identically imply the existence of more, thus reducing the number of distinct cases to be covered). Nevertheless, to see this project through to completion will clearly involve a massive computational effort.

In [35], we have begun with the case of a Spin(7) structure, and have derived the constraints implied by imposing

$$[\mathcal{D}_\mu, g\Gamma^-]\epsilon = 0. \tag{75}$$

We have found that the the general bosonic solution of the Killing spinor equation admitting a Spin(7) structure is determined locally as follows. We may take the Killing spinors to be ϵ, $H^{-1/3}(x)\Gamma^-\epsilon$, with metric

$$\begin{aligned} ds^2 &= H^{-2/3}(x)\Big(2[du + \lambda(x)_M dx^M][dv + \nu(x)_N dx^N] + [dz + \sigma(x)_M dx^M]^2\Big) \\ &+ H^{1/3}(x) h_{MN}(x) dx^M dx^N, \end{aligned} \tag{76}$$

where h_{MN} is a metric of Spin(7) holonomy and $d\lambda$, $d\nu$ and $d\sigma$ are two-forms in the **21** of Spin(7). Observe that there are three Killing vectors. Defining the elfbeins

$$\begin{aligned} e^+ &= H^{-2/3}(du+\lambda), \\ e^- &= dv+\nu, \\ e^9 &= H^{-1/3}(dz+\sigma), \\ e^i &= H^{1/6}\hat{e}^i(x)_M dx^M, \end{aligned} \tag{77}$$

where $\hat{e}^i$ are the achtbeins for h, the four-form is

$$\begin{aligned} F &= e^+ \wedge e^- \wedge e^9 \wedge d\log H + H^{-1/3} e^+ \wedge e^- \wedge d\sigma - e^+ \wedge e^9 \wedge d\nu \\ &+ H^{-2/3} e^- \wedge e^9 \wedge d\lambda + \frac{1}{4!} F^{\mathbf{27}}_{ijkl} \hat{e}^i \wedge \hat{e}^j \wedge \hat{e}^k \wedge \hat{e}^l. \end{aligned} \tag{78}$$

We expect that classifying all supersymmetric spacetimes with G-structures whose structure groups fix at most eight Killing spinors will be eminently practical. Considerably more effort will be required to completely classify the cases ($\mathbb{R}^9$ and $SU(2)$) where the structure group fixes at most sixteen. Finally, the classification of all spacetimes admitting an Identity structure will be, without further insight, technically difficult. This is because the Killing spinors in this case can be completely generic, and the constraints implied by the existence of a completely generic Killing spinor will be very complicated. It would be very useful to have a complementary "top down" formalism, whereby one could start with the constraints for maximal supersymmetry, and progressively and systematically weaken them, for the classification of spacetimes admitting an Identity structure and preserving $N > 16$. Nevertheless, we believe that classifying all spacetimes with an Identity structure will be a hard problem, irrespective of the approach employed.

Clearly, the formalism we have used is universally applicable. Simpler lower dimensional supergravites (particularly those with eight supercharges) should be easy to analyse. It should also be possible to apply it to IIB, though given the formal complexity of this theory, the amount of computation required to perform the complete classification will, if anything, be greater than that in eleven dimensions (though there is the simplification that in IIB the spinors are always null [16], and all structure groups are subgroups of $Spin(7) \ltimes \mathbb{R}^8$).

Of course, categorizing the general local bosonic solution of the Killing spinor equation of eleven dimensional supergravity is not the same as determining all the supersymmetric solutions of the field equations and Bianchi identity; some subset thereof will still need to be imposed. Nevertheless, it is our hope that the procedure we are advocating will ultimately lead to the construction of many new solutions, from which much will be learned about the physics of supersymmetric spacetimes in M-theory.

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